

# Two-dimensional Quantum Field Models (with applications to Statistical Mechanics)

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## Abstract

Two dimensional toy models display, in a gentler setting, many salient aspects of QFT. I discuss a concrete two dimensional case, the Thirring model, which, in spite of its simple definition, illustrates some celebrated features of QFT: the anomalous dimension of the fields; the exact solvability; the anomalies of the Ward-Takahashi identities. Besides, I give a glimpse of the crucial role that this model plays in the study of something apparently quite different: two dimensional, lattice systems of Statistical Mechanics.

*Keywords* Osterwalder-Schrader axioms, exact solutions, bosonization, Ward-Takahashi identities, anomalies, Ising model, Eight-Vertex model, Ashkin-Teller model

## 1 Introduction

The Thirring model, [1], is a toy model for a self-interacting, 2-dimensional, relativistic fermion field. Over the years, an enormous number of physicists' papers testified its importance in the study of the general conjectures which have been proposed for QFT models. In the first part of this review I will deal with the two most salient features of this model: the *bosonization* of the currents; and, in the massless case, the *exact solvability*. In doing so, the discussion of the mathematical results will follow the presentation of the physicists' conjectures and ansatz.

In the second part of the article, I will deal with a third, less known, aspect of the Thirring model, which is though as important as the previous ones: the Thirring model is the *scaling limit* of several lattice models of Statistical Mechanics. Recently, the solution of important mathematical problems have been obtained along this direction.

## 2 Thirring Model

Let  $\{\gamma^\mu\}_{\mu=0,1}$  be 2 by 2 matrices which generate the two dimensional Euclidean Clifford algebra; let  $\not{\partial} := \gamma^\mu \partial^\mu$  (with sum on repeated indexes) be the two-dimensional Dirac operator; and let  $\bar{\psi}_{\mathbf{x}} = \begin{pmatrix} \bar{\psi}_{\mathbf{x},+} \\ \bar{\psi}_{\mathbf{x},-} \end{pmatrix}$  and  $\psi_{\mathbf{x}} = (\psi_{\mathbf{x},+}, \psi_{\mathbf{x},-})$ , where  $\mathbf{x} := (x_0, x_1)$ , be two-components anticommuting vector fields. Given two real parameters,  $\lambda$ , the coupling constant, and  $m$ , the mass, the (Euclidean) path integral formulation of the Thirring model has Lagrangian

$$\int d^2 \mathbf{x} \bar{\psi}_{\mathbf{x}} (i \not{\partial} + m) \psi_{\mathbf{x}} + \frac{\lambda}{4} \int d^2 \mathbf{x} (\bar{\psi}_{\mathbf{x}} \gamma^\mu \psi_{\mathbf{x}})^2$$

Let us consider separately the exact solvability and the bosonization.

### 2.1 Massless Thirring Model: the formal exact solution

The sequence of early attempts at an exact solution of the massless model is quite an instructive piece of history of QFT, [2]. At first, several scholars “derived” different sets of explicit formulas for all the Schwinger functions; but “... it was not clear which of the many partly contradictory equations written by the various authors were true and which false.” (in [2]). In this confusing status, the model fell into disrepute. Until Johnson, [3], was finally able to propose formulas for the two and the four point functions that were apparently not susceptible to objections. Johnson’s two point function is

$$\frac{1}{Z} \langle \psi_{\mathbf{x}} \bar{\psi}_0 \rangle = C \begin{pmatrix} 0 & \frac{1}{x_0 + ix_1} |\mathbf{x}|^{-\eta} \\ \frac{1}{x_0 - ix_1} |\mathbf{x}|^{-\eta} & 0 \end{pmatrix} \quad (2.1)$$

where  $C$  and  $\eta$  are constants; while  $Z$  is the *wave function renormalization* (i.e. an infinite or zero factor that one has to divide out in the end of the derivation of the Schwinger functions in order to get a finite, non-identically vanishing, outcome; in this model,  $Z \sim e^{-\eta \cdot \infty}$  and  $\eta = O(\lambda)$ ).

Johnson’s approach was only partially a derivation; it was mostly a self-consistency argument. The fundamental idea is that (2.1) should be the solution of the integral equation which is obtained by plugging the vector and the axial-vector Ward-Takahashi Identities (WTI) into the Schwinger-Dyson equation. If  $J_{\mathbf{x}}^\mu := \bar{\psi}_{\mathbf{x}} \gamma^\mu \psi_{\mathbf{x}}$  and  $J_{5,\mathbf{x}}^\mu := \bar{\psi}_{\mathbf{x}} \gamma^5 \gamma^\mu \psi_{\mathbf{x}}$  for  $\gamma^5 = i\gamma^0 \gamma^1$ , the *formal* WTI are

$$\frac{1}{Z} i \partial_{\mathbf{z}}^\mu \langle J_{\mathbf{z}}^\mu \psi_{\mathbf{x}} \bar{\psi}_{\mathbf{y}} \rangle = \frac{a}{Z} \left[ (\mathbf{z} - \mathbf{x}) - (\mathbf{z} - \mathbf{y}) \right] \langle \psi_{\mathbf{x}} \bar{\psi}_{\mathbf{y}} \rangle$$

$$\frac{1}{Z} i \partial_{\mathbf{z}}^{\mu} \langle J_{5,\mathbf{z}}^{\mu} \psi_{\mathbf{x}} \bar{\psi}_{\mathbf{y}} \rangle = \frac{\bar{a}}{Z} \left[ (\mathbf{z} - \mathbf{x}) - (\mathbf{z} - \mathbf{y}) \right] \gamma^5 \langle \psi_{\mathbf{x}} \bar{\psi}_{\mathbf{y}} \rangle \quad (2.2)$$

with  $a = \bar{a} = 1$ . But then, after a computation, one would obtain (2.1) with  $\eta = 0$ ; namely the Thirring model, that is an interacting field theory, would have the same two point function of the free field theory! Johnson's idea was to allow for  $a \neq 1$  and  $\bar{a} \neq 1$ , so that, after *formal operations with infinities*, he obtained (2.1), with

$$\eta = \frac{\lambda}{4\pi} (a - \bar{a}) .$$

Then he determined  $a$  and  $\bar{a}$  by consistency with the formula for the four point function: it turned out, setting  $\nu = -\bar{\nu} = \frac{\lambda}{4\pi}$ ,

$$a = \frac{1}{1 - \nu} \quad \bar{a} = \frac{1}{1 - \bar{\nu}} \quad (2.3)$$

Subsequently, Hagen, [4], (by Johnson's method) and Klaiber, [5], (with an approach that is an ansatz in even more explicit terms), found the explicit formula for all the  $n$ -points Schwinger functions.<sup>1</sup> The only non-zero ones are

$$\frac{1}{Z^n} \langle \psi_{\mathbf{x}_1, \omega_1} \cdots \psi_{\mathbf{x}_n, \omega_n} \bar{\psi}_{\mathbf{y}_1, \sigma_1} \cdots \bar{\psi}_{\mathbf{y}_n, \sigma_n} \rangle = \sum_{\pi} (-1)^{\pi} G_{\underline{\omega}, \pi(\underline{\sigma})}(\underline{\mathbf{x}}, \pi(\underline{\mathbf{y}})) \quad (2.4)$$

where  $\pi$  runs over the permutations of  $n$  elements and  $(-1)^{\pi}$  is the signum of the permutation; and for  $\omega_j = \pm 1$  and  $\sigma_j = \pm 1$ ,

$$G_{\underline{\omega}, \underline{\sigma}}(\underline{\mathbf{x}}, \underline{\mathbf{y}}) := \frac{1}{Z} \langle \psi_{\mathbf{x}_1, \omega_1} \bar{\psi}_{\mathbf{y}_1, \sigma_1} \rangle \cdots \frac{1}{Z} \langle \psi_{\mathbf{x}_n, \omega_n} \bar{\psi}_{\mathbf{y}_n, \sigma_n} \rangle \cdot \frac{\prod_{i < j} |\mathbf{x}_i - \mathbf{x}_j|^{\eta_- \omega_i \omega_j} \prod_{i < j} |\mathbf{y}_i - \mathbf{y}_j|^{\eta_- \sigma_i \sigma_j}}{\prod_{i \neq j} |\mathbf{x}_i - \mathbf{y}_j|^{\eta \omega_i \sigma_j}} \quad (2.5)$$

where  $\eta_- = \eta$  and  $\eta_+$  is a new coefficient.

Are these solutions completely satisfactory? As we saw, in no case they were obtained by rigorous procedures. Yet Wightman, [2], pointed out that if the  $n$ -points functions satisfy some suitable set of axioms, say the Osterwalder-Schrader's ones, then one is entitled to claim to have constructed a QFT! In this case, surprisingly enough, although one has explicit formulas, it has not been an easy task to verify an axiom in particular, the reflection positivity. I will return on this matter in the section on mathematical results. Let me emphasize here that this viewpoint is applicable, of course, only in models for which there exists an ansatz of exact solution.

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<sup>1</sup>In fact they also generalize Johnson's solution, since they obtained  $\nu = \frac{\lambda}{2\pi}(1 - \xi)$  and  $\bar{\nu} = -\frac{\lambda}{2\pi}\xi$ , for any real  $\xi$ . Several years later  $\nu$  and  $\bar{\nu}$  would have been called Adler-Bell-Jackiw anomalies of the vector and the axial-vector WTI;  $\xi$  is regularization dependent: in the dimensional regularization  $\xi = 1$  and only the axial-vector WTI is anomalous.

## 2.2 General Thirring Model: Bosonization

Let us consider, now, two different models. The first is the Thirring model with a possible mass term. The second is a boson system, called *sine-Gordon* model: for two real parameters  $\beta > 0$  and  $z$ , the Lagrangian is

$$\frac{1}{2\beta} \int d^2 \mathbf{x} (\partial^\mu \phi_{\mathbf{x}})^2 + z \int d^2 \mathbf{x} : \cos \phi_{\mathbf{x}} :$$

where  $:\cos \phi_{\mathbf{x}}:$  is the *normal ordering* of  $\cos \phi_{\mathbf{x}}$  (which corresponds to multiply  $\cos \phi_{\mathbf{x}}$  by an infinite factor). Coleman, [6], discovered a surprising relation between the two models. Define the fermion currents

$$J_{\mathbf{x}}^\mu := \bar{\psi}_{\mathbf{x}} \gamma^\mu \psi_{\mathbf{x}} \quad O_{\mathbf{x}}^\sigma := \bar{\psi}_{\mathbf{x}} (1 + \sigma \gamma^5) \psi_{\mathbf{x}}$$

(for  $\mu = 0, 1$  and  $\sigma = \pm 1$ ); and define the boson observables

$$\mathcal{J}_{\mathbf{x}}^\mu := -\frac{1}{\sqrt{\pi}} \epsilon^{\mu\nu} \partial_\nu \phi_{\mathbf{x}} \quad \mathcal{O}_{\mathbf{x}}^\sigma := e^{i\sigma\phi_{\mathbf{x}}} :$$

where  $:e^{i\sigma\phi_{\mathbf{x}}}: equals  $e^{i\sigma\phi_{\mathbf{x}}}$  times an infinite factor. The bosonization claim is that there exists a choice of  $\beta$  and  $z$  as function of  $\lambda$  and  $m$ , with  $\beta = 4\pi + O(\lambda)$  and  $z = O(m)$ , such that, if  $\mathbf{x}_1, \dots, \mathbf{x}_n, \mathbf{y}_1, \dots, \mathbf{y}_m$  are two by two different,$

$$\zeta_J^n \zeta_O^m \langle J_{\mathbf{x}_1}^{\mu_1} \dots J_{\mathbf{x}_n}^{\mu_n} O_{\mathbf{y}_1}^{\sigma_1} \dots O_{\mathbf{y}_m}^{\sigma_m} \rangle_{Th} = \langle \mathcal{J}_{\mathbf{x}_1}^{\mu_1} \dots \mathcal{J}_{\mathbf{x}_n}^{\mu_n} \mathcal{O}_{\mathbf{y}_1}^{\sigma_1} \dots \mathcal{O}_{\mathbf{y}_m}^{\sigma_m} \rangle_{sG}$$

where  $\zeta_J$  and  $\zeta_O$  are possibly infinite or zero prefactor, needed to have finite, non-zero correlations of local products of fermions; and  $\langle \cdot \rangle_{Th}$  and  $\langle \cdot \rangle_{sG}$  are the Thirring and the sine-Gordon expectations in the sense of path-integrals. Notice two facts. First, for  $m = 0$  one has  $z = 0$ , therefore the current correlations of the massless Thirring model equal *free boson correlations*. Second, for  $m \neq 0$  the correlation of the Thirring model are expected to decay exponentially; and so has to happen then to the boson correlations, even though the sine-Gordon model does not have a mass term in the Lagrangian: this hints a *dynamically generated mass* for the boson field. Finally let me remark that the bosonization at  $m = 0$  was known much earlier in condensed matter physics, [7, 8]; while the general Coleman's argument was made more precise by Mandelstam, [9].

## 2.3 Some Mathematical Results on the Thirring Model

For the massless Thirring model the first (and, for a long while, the only) mathematical result was the proof, [10], of the reflection positivity of the Hagen's and Klaiber's formulas (the fulfillment of the other Osterwalder-Schrader's axioms was evident).

In regard to the general Thirring model and the bosonization, the first mathematical work was [11]. It contained the proof of analyticity in  $z$  of the sine-Gordon model for  $|z|$  small enough and  $\beta \in [0, 4\pi)$ . This was an important achievement, for the Coleman's conjecture was based on the identity of the coefficients of the perturbation theories of the Thirring and of the sine-Gordon

models. But it was not a *proof* of bosonization, because the authors could *not* deal with the perturbation theory of the Thirring model as well. Their approach was based on the Glimm-Jaffe-Spencer renormalization group (RG) method, [12], which was the first rigorous implementation of the Wilson's ideas - and a milestone in this subject - but had the limitation of being applicable only if the model is *super-renormalizable*. This is the case of the sine-Gordon model, for  $\beta \in [0, 4\pi)$ ; but it is not the case of the Thirring model, which is instead *renormalizable* and so requires a more sophisticated treatment. (Besides, to avoid the mathematical difficulty of the spontaneous mass generation, in [11] a mass term  $\mu^2 \int d^2\mathbf{x} \phi_{\mathbf{x}}^2$  is built in the Lagrangian of the sine-Gordon model, which then corresponds to the bosonization of a slightly different fermion system, called Schwinger-Thirring model). After some years, Dimock, [13], using the RG approach of Brydges and Yau, [14], extended the result of [11] to  $\beta \in [0, \frac{16}{3}\pi)$ . (He avoided the problem of the spontaneous mass generation, by confining the boson interaction term to a finite volume). Anyways, he still had no result for the Thirring model.

Some of the problems that were left open by these early papers were settled in a series of results in collaborations with G. Benfatto and V. Mastropietro. Starting from the path-integral formulation of the Thirring model, regularized by the presence of an ultraviolet and an infrared cutoffs, we proved the following facts, [15, 16, 17].

- *Massless case.* For  $|\lambda|$  small enough, there exists the limit of removed cutoffs of all the  $n$ -point Schwinger functions. Such limiting functions satisfy the Osterwalder-Schrader axioms, and coincide with the Hagen's and Klaiber's ansatz (2.4). Besides, bosonization is proved for  $|\beta - 4\pi|$  small enough.
- *Massive case.* For any  $m$  and  $|\lambda|$  small enough, again there exists the limit of removed cutoffs of all the  $n$ -point Schwinger functions. The Osterwalder-Schrader axioms are fulfilled. Besides, bosonization is proved for  $|\beta - 4\pi|$  small enough *if* both the fermion mass term and the boson interaction term are *confined to a finite volume* (to overcome the difficulty of the spontaneous mass generation).

The major problem that these works leave open, then, is the proof of bosonization when the spontaneous mass generation is not artificially avoided. Anyways, to the best of my knowledge, these are the first rigorous results for the massive Thirring model. An also for the massless one, this was the first time the exact solution is *derived* from the regularized path-integral formulation: this fact will be of crucial importance for the applications to Statistical Mechanics. These works are based on the RG technique developed by Gallavotti and Nicoló, [18, 19] and on a technical result called *vanishing of the beta function*, [20, 21]. To understand the difficulty of the problem, let us analyze some aspects of the results. From the second line of (2.5) we see that, in the  $n$ -point functions, the decay for large separation of the points is not the same as in the products of two point functions. Besides, the two point function itself does not have the same

large distance decay of the free field, but displays an *anomalous exponent*,  $\eta$ . In the jargon of the Renormalization Group, the Thirring model is said to describe a *Non-Gaussian Fixed Point* (in fact, since the value of  $\eta$  depends upon  $\lambda$ , here we have a case of an *interval* of non-Gaussian fixed points).

Let me also emphasize some differences w.r.t. Johnson's work. (i) Of course, we do not work with infinities, but the theory is regularized and the cutoffs are removed in the final results only. On the same time, we do not modify by hands the formal WTI; the coefficients  $\nu \neq 0$  and  $\bar{\nu} \neq 0$  naturally arise in the limit of removed cutoffs from terms that, in formal treatments, are considered vanishing. That is known to physicists as Adler-Bell-Jackiw mechanism, [22, 23]. (ii) Our  $\nu$  and  $\bar{\nu}$  are not linear in  $\lambda$ ; besides, our value for  $\frac{\eta}{a-a}$  is not linear in  $\lambda$  either<sup>2</sup>. Point (ii) does not mean that Johnson's solution is wrong. It is a general expectation in QFT that macroscopic quantities, such as  $\eta$ , are related to the bare parameters in the Lagrangian, such as  $\lambda$ , in a regularization-dependent way. Indeed, as counter-proof, a different regularization of the Thirring model (a "non-local" one) does give  $\nu$ ,  $\bar{\nu}$  and  $\frac{\eta}{a-a}$  linear in  $\lambda$ , [24]. That is known to physicists as Adler-Bardeen theorem, [25].

Our RG approach has other applications in QFT. In two cases, exact solutions (previously conjectured by physicists) were rigorously derived from the regularized path-integral formulation of the models: for the Thirring-Wess model (i.e. a fermion field interacting with a vector model), see [26]; and for a two colors generalization of the massless Thirring model (which can include also an interaction that is not rotational invariant), see [27]. For lack of space, I cannot describe the details.

### 3 Lattice models of Statistical Mechanics

The most basic lattice model of two-dimensional Statistical Mechanics is the square lattice Ising model with finite range interactions. In particular, let us consider the following Hamiltonian: for real  $J$  and  $K$  (and spin  $\sigma_{\mathbf{x}} = \pm 1$ ),

$$H(\underline{\sigma}) := -J \sum_{\substack{\mathbf{x}, \mathbf{x}' \\ \text{n.n.}}} \sigma_{\mathbf{x}} \sigma_{\mathbf{x}'} - K \sum_{\substack{\mathbf{x}, \mathbf{x}' \\ \text{n.n.n.}}} \sigma_{\mathbf{x}} \sigma_{\mathbf{x}'}$$

where the first sum is over nearest-neighbor (n.n.) sites and the second in over next nearest-neighbor (n.n.n.) sites. The case  $K = 0$  is the celebrated one, the one for which Onsager discovered the non-trivial exact formula of the free energy, [28]. Instead, providing any rigorous result for  $K \neq 0$  remained for many years as an open problem. Until very recently, when Spencer, [29], proposed how to (rigorously) calculate certain critical exponents at  $K \neq 0$ . To understand his idea, it is important to remark at this point that the reason behind the exact solvability at  $K = 0$  is the equivalence of the Ising model with a system of *free*

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<sup>2</sup>interesting enough, this fact is due to a "new anomaly" which is tightly related to the exact solvability of the model whenever a "local" regularization is employed, [15]

lattice fermions, [30, 31, 32]. Spencer's suggestion is to use the same fermion rephrasing also in the non-solvable case. Of course the lattice fermion fields, this time, are not free, but self-interacting; yet, handled by RG, the self-interaction turns out to be an *irrelevant* perturbation of the free field. As consequence, if  $|K|$  is small enough w.r.t.  $|J|$ , the critical exponent should remain unchanged. Following these ideas, Pinson and Spencer proved the following result, [33]. Let

$$O_{\mathbf{x}} := \sum_{\substack{\mathbf{x}' \\ \text{n.n. of } \mathbf{x}}} \sigma_{\mathbf{x}} \sigma_{\mathbf{x}'}$$

then, if  $|K|$  is small enough w.r.t.  $|J|$ , there exists one ( $J$  and  $K$  dependent) critical temperature at which, asymptotically for large  $|\mathbf{x}|$ ,

$$\langle O_{\mathbf{x}} O_0 \rangle \sim \frac{C}{|\mathbf{x}|^{2\kappa_+}} \quad \text{for } \kappa_+ = 1$$

The fact that the *energy critical exponent*,  $\kappa_+$ , is independent of  $K$  (and coincides with the one of the n.n. Ising model) is called *universality*.

This approach to the criticality, which from now on I will call *interacting fermions picture* (IFP), has a much wider applicability. Consider the class of models made of two (apriori independent) n.n. Ising models that are connected to each others by a quartic interaction. More precisely, models with Hamiltonian

$$H(\underline{\sigma}, \underline{\tau}) := -J \sum_{\substack{\mathbf{x}, \mathbf{x}' \\ \text{n.n.}}} \sigma_{\mathbf{x}} \sigma_{\mathbf{x}'} - J \sum_{\substack{\mathbf{x}, \mathbf{x}' \\ \text{n.n.}}} \tau_{\mathbf{x}} \tau_{\mathbf{x}'} + K \sum_{\substack{\mathbf{x}, \mathbf{x}' \\ \text{n.n.}}} \sum_{\substack{\mathbf{y}, \mathbf{y}' \\ \text{n.n.}}} \sigma_{\mathbf{x}} \sigma_{\mathbf{x}'} v(\mathbf{x} - \mathbf{y}) \tau_{\mathbf{y}} \tau_{\mathbf{y}'}$$

where  $\sigma_{\mathbf{x}} = \pm 1$  and  $\tau_{\mathbf{x}} = \pm 1$  are two spins located at the same site; and  $|v(\mathbf{x})| \leq C e^{-c|\mathbf{x}|}$ . The role played by this class of *double Ising models* (DIM) is not merely academic, since it encompasses two lattice systems that are famous for historical and technical reasons: the Ashkin-Teller and the Eight Vertex models<sup>3</sup>. Of course for  $K = 0$  the model is again exactly solvable (and called *free fermion point*). And for  $K \neq 0$  one can derive an equivalence with a self-interacting lattice fermions. The novelty w.r.t. the previous case is that the self-interaction is now *marginal*, and so it does change the large distance decay of the observables. The following facts are proven in [35]. Define the *energy* and the *crossover* observables to be

$$O_{\mathbf{x}}^+ := \sum_{\substack{\mathbf{x}' \\ \text{n.n. of } \mathbf{x}}} \sigma_{\mathbf{x}} \sigma_{\mathbf{x}'} + \sum_{\substack{\mathbf{x}' \\ \text{n.n. of } \mathbf{x}}} \tau_{\mathbf{x}} \tau_{\mathbf{x}'} \quad O_{\mathbf{x}}^- := \sum_{\substack{\mathbf{x}' \\ \text{n.n. of } \mathbf{x}}} \sigma_{\mathbf{x}} \sigma_{\mathbf{x}'} - \sum_{\substack{\mathbf{x}' \\ \text{n.n. of } \mathbf{x}}} \tau_{\mathbf{x}} \tau_{\mathbf{x}'}$$

Then, if  $|K|/|J|$  is small enough, there exists one ( $J$  and  $K$  dependent) critical temperature at which, for large separation  $|\mathbf{x}|$ ,

$$\langle O_{\mathbf{x}}^+ O_0^+ \rangle \sim \frac{C_+}{|\mathbf{x}|^{2\kappa_+}} \quad \langle O_{\mathbf{x}}^- O_0^- \rangle \sim \frac{C_-}{|\mathbf{x}|^{2\kappa_-}}$$

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<sup>3</sup>exact formulas for the free energies of these two models (but not, in general, for all the DIM) are available, [34]. Though, to avoid confusion, I will not call them exactly solvable model for, as opposed to what happens in the n.n. Ising model, no exact formula is known for correlation functions.

with  $\kappa_+ \equiv \kappa_+(\lambda, v) = 1 + O(\lambda)$ ,  $\kappa_- \equiv \kappa_-(\lambda, v) = 1 + O(\lambda)$  and  $\lambda = \frac{K}{J}$ . From a mathematical viewpoint this result is certainly interesting, because there was no result at all for the correlations of any of the DIM. But is this result physically significant? As opposed to the case of the Ising model,  $\kappa_+$  and  $\kappa_-$ , which are macroscopic quantities, do depend upon  $\lambda$  and  $v(\mathbf{x})$ , which are parameters in the definition of the toy model. This means that the DIM class is *non-universal*. Anyways, Kadanoff discovered that a weak form of universality still persists: on the basis of the fact that the scaling limit of this class of models turns out to be the Thirring model (!), he predicted, [36], the formula

$$\kappa_+(\lambda, v) \cdot \kappa_-(\lambda, v) = 1$$

The Kadanoff's formula is now proven, [37]. The idea of the proof is that the regularized path-integral formulation of the Thirring model and the fermion phrasing of the partition function of the DIM differ by irrelevant terms. Since large distances make irrelevant interactions negligible, as  $\kappa_+ \cdot \kappa_- = 1$  is satisfied in the exact solution of the massless Thirring model, it has to hold also in the lattice model<sup>4</sup>. Let me emphasize that in this chain of implications the only knowledge of the Hagen's and Klaiber's formulas would not suffice: one really needs a mathematical derivation of the Schwinger functions from the regularized path-integral formula!

There are several other cases in which the IFP is or should be resolute. In [17, 38, 27] we proved similar formulas, called "Luttinger Liquid Relations" for the XYZ quantum chain and for a generalization of the  $(1+1)$ -dimensional Hubbard model. In prospective, I think that the study of critical exponents of local bulk observables of *weakly interacting dimers* (on square or hexagonal lattices, for instance) and of the *six vertex model* (close enough to the free fermion point) should be feasible by IFP.

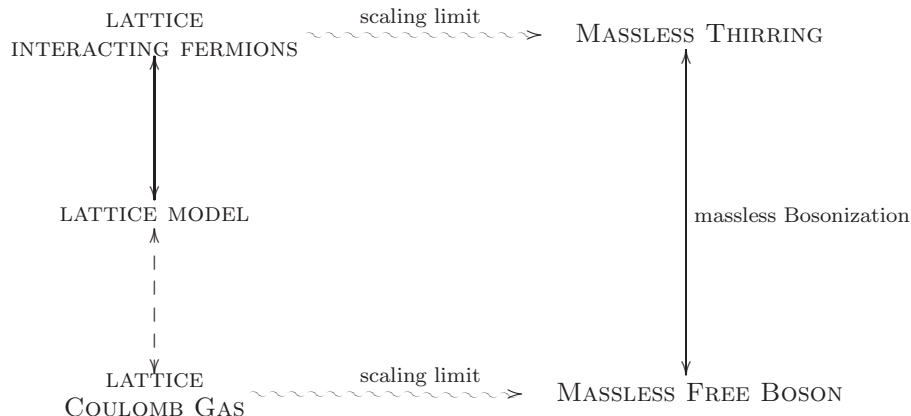
## 4 Conclusions

What I have discussed so far is only a part of a general picture that the physicists of the 70's and 80's discovered, and which is briefly depicted in the following diagram.

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<sup>4</sup>although with (in general) a different dependence of  $\kappa_+$  and  $\kappa_-$  upon  $\lambda$





The task is to compute critical exponents of lattice models (center-right of the diagram). To achieve that, it is useful to compute the scaling limit (i.e. the continuum limit) of such models, which has more chances of being exactly solvable. There are two ways of doing that. The one explained in this review is to use the IFP, the scaling limit of which is the Thirring model (upper part of the diagram). But also another approach is possible. I have no space to explain the details, but basically it consists in re-casting the lattice model into a *lattice Coulomb gas*, the scaling limit of which is the free boson field. The agreement of the two different approaches is provided by the bosonization. The upper route, proposed by many, including Kadanoff, [36], and den Nijs, [39], has been made mathematically rigorous in some models: n.n. Ising with n.n.n. perturbation, [33], the XYZ quantum chain, [37], a generalization of the Hubbard model, [27], and the class of DIM, [37]. Hopefully it can be applied to the weakly interacting dimers and to the six vertex model. The lower route was introduced by Kadanoff, [40], Nienhuis, [41], and others. Some of the conjectures about the lattice Coulomb gas are now proven, [42, 43]; but a rigorous implementation of the equivalence lattice model / Coulomb gas (the broken line in the diagram) is unfortunately missing.

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